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Locally dually flat Finsler metrics with special curvature properties ☆

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Locally dually flat Finsler metrics are studied in Finsler information geometry and naturally arise from the investigation of the so-called flat information structure. In this survey article, we first characterize locally dually flat and projectively flat Finsler metrics. Then we mainly study locally dually flat Randers metrics in the form $F=\alpha+\beta$, where α is a Riemannian metric and β is a 1-form on the manifold. We find some equations that characterize locally dually flat Randers metrics and classify locally dually flat Randers metrics with weak isotropic flag curvature. Further, we characterize locally dually flay (α,β)-metrics in the form $F=\alpha+\epsilon\beta+k\frac{\beta^2}{\alpha}$ of scalar flag curvature, where ϵ, k are nonzero constants.

1. Definitions and notations

A Finsler metric on a C^∞ manifold M is a continuous function $F: TM \to [0,\infty)$ with the following properties:

(1) *Regularity*: $F(x,y)$ is C^∞ on $TM\backslash\{0\}$.
(2) *Homogeneity*: $F(x,\lambda y)=\lambda F(x,y)$, $\forall\lambda>0$.
(3) *Strong convexity*: the fundamental tensor $(g_{ij}(x,y))$ is positive definite, where

$$g_{ij}(x,y):=\frac{1}{2}\left[F^2\right]_{y^iy^j}(x,y).$$

A Finsler metric $F=F(x,y)$ on M is said to be reversible if, at each point $x\in M$,

$$F(x,-y)=F(x,y),\quad \forall y\in T_xM.$$

In a Finsler manifold (M,F), for each $y\in T_xM\backslash\{0\}$, we can define an inner product $g_y: T_xM\times T_xM\to R$ in T_xM as follows

$$g_y(u,v)=g_{ij}(x,y)u^iu^j,\qquad u=u^i\frac{\partial}{\partial x^i}\bigg|_x,\qquad v=v^j\frac{\partial}{\partial x^j}\bigg|_x\in T_xM.$$

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By the homogeneity of F,

$$F(x,y)=\sqrt{g_{ij}(x,y)y^iy^j}.$$

Remark 1.1. The following are some special Finsler metrics.

(1) *Riemann metric.* $F(x,y)=\sqrt{g_{ij}(x)y^iy^j}$. In this case, g_{ij} are independent of the direction y. Thus the fundamental tensor for general Finsler metrics may be thought of as a direction dependent Riemannian metric.
(2) *Minkowski metric.* $F(x,y)=\sqrt{g_{ij}(y)y^iy^j}$. In this case, F is independent of the position x.
(3) *Randers metric.* $F=\alpha+\beta$, where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemann metric and $\beta=b_i(x)y^i$ is a 1-form with

$$\|\beta\|_\alpha(x):=\sqrt{a^{ij}(x)b_i(x)b_j(x)}<1,\quad \forall x\in M.$$

Randers metrics was introduced by physicist G. Randers in 1941 [18] from the standpoint of general relativity, who used α to denote gravitation field and used β to denote electromagnetic field. Later on, these metrics were applied to the theory of the electron microscope by R.S. Ingarden in 1957, who first named them Randers metrics. Now, Randers metrics have formed an important class of Finsler metrics.

Finsler spray induced by Finsler metric F is a tangent vector field globally defined on TM defined by

$$\mathbf{G}:=y^i\frac{\partial}{\partial x^i}-2G^i(x,y)\frac{\partial}{\partial y^i},$$

where

$$G^i(x,y):=\frac{1}{4}g^{il}\big\{\big[F^2\big]_{x^my^l}y^m-\big[F^2\big]_{x^l}\big\},$$

where $(g^{ij}(x,y)):=(g_{ij}(x,y))^{-1}$. We call G^i the *geodesic spray coefficients* of F. G^i are so named because the geodesics of F are the solutions of the differential equations

$$\ddot{x}^i+2G^i(x,\dot{x})=0.$$

When (M,F) is Riemannian manifold, we have

$$G^i(x,y)=\frac{1}{2}\Gamma^i_{jk}(x)y^iy^j,$$

where $\Gamma^i_{jk}(x)$ denote the Christoffel symbols.

The notion of Riemann curvature for Riemann metrics can be extended to Finsler metrics. For a vector $y\in T_xM\setminus\{0\}$, the *Riemann curvature* $\mathbf{R}_y:T_xM\to T_xM$ is defined by

$$\mathbf{R}_y(u):=R^i_k(x,y)u^k\frac{\partial}{\partial x^i}\bigg|_x,\qquad u=u^i\frac{\partial}{\partial x^i}\bigg|_x,$$

where

$$R^i_k(x,y):=2\frac{\partial G^i}{\partial x^k}-y^j\frac{\partial^2G^i}{\partial x^j\partial y^k}+2G^j\frac{\partial^2G^i}{\partial y^j\partial y^k}-\frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{1}$$

In fact, Riemann curvature naturally arises from the geodesic variation of geodesics.

Take an arbitrary plane $P\subset T_xM$ (flag) and a nonzero vector $y\in P$ (flag pole), the *flag curvature* $\mathbf{K}(x,y,P)$ is defined by

$$\mathbf{K}(x,y,P):=\frac{g_y(\mathbf{R}_y(u),u)}{g_y(y,y)g_y(u,u)-[g_y(u,y)]^2}, \tag{2}$$

where u is an arbitrary vector in P such that $P=span\{y,u\}$.

When F is Riemannian, $\mathbf{K}(x,y,P)=\mathbf{K}(x,P)$ is independent of y. In this case, $\mathbf{K}$ is just the sectional curvature.

F is said to be *of scalar flag curvature* if $\mathbf{K}(x,y,P)=\mathbf{K}(x,y)$ is independent of P. F is said to be of *weak isotropic flag curvature* if $\mathbf{K}(P,y)=\frac{3c_{x^m}y^m}{F}+\sigma$, where $c=c(x)$ and $\sigma=\sigma(x)$ are scalar functions on M. In particular, F is said to be of *constant flag curvature* if $\mathbf{K}(P,y)=\sigma$, where σ is constant.

A Finsler metric F is called *projectively flat* if F is projectively equivalent to a Minkowski/Euclidean metric. In this case, all geodesics of F are straight lines, namely, we can characterize geodesics of F as $\sigma(t):=f(t)a+b$ for some constant vectors $a,b\in R^n$.

Lemma 1.2. *(See [13].) A Finsler metric F on a manifold M is projectively flat if and only if F satisfies the following*

$$F_{x^k y^l} y^k - F_{x^l} = 0. \tag{3}$$

In this case,

$$G^i = P(x, y) y^i \tag{4}$$

with

$$P = \frac{F_{x^k} y^k}{2F}.$$

We call P the projective factor of F.

In regular case, the problem of characterizing and studying projectively flat Finsler metrics is known as *Hilbert's Fourth Problem*. Hilbert's Fourth Problem is to characterize the distance functions on an open subset in R^n such that straight lines are the shortest paths. Projectively flat Finsler metrics are just the smooth solutions of Hilbert's Fourth Problem in the regular case.

A fundamental fact is that any projectively flat Finsler metric must be of scalar flag curvature with the following form,

$$\mathbf{K} = \frac{P^2 - P_{x^m} y^m}{F^2}. \tag{5}$$

By [3], a Randers metric $F = \alpha + \beta$ is projectively flat if and only if α is projectively flat and β is closed. For Riemannian metrics, Beltrami Theorem says that a Riemannian metric is projectively flat if and only if it is of constant sectional curvature [20]. A natural question arises: whether or not is Beltrami Theorem still true in Finsler geometry? Unfortunately, the answer is negative. The first example was given by D. Bao and Z. Shen in [6]. They constructed a Randers metric on the Lie group S^3 which is of constant flag curvature $\mathbf{K} = 1$ but is not projectively flat.

In Finsler geometry, there are some important geometric quantities which have many important influences on the geometric structures of Finsler metrics and vanish in Riemannian case. We call them *non-Riemannian quantities*.

In order to characterize Riemannian metrics in Finsler metrics, we first introduce Cartan torsion $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbf{R}$, which is defined by

$$\mathbf{C}_y(u, v, w) := C_{ijk}(x, y) u^i v^j w^k,$$

where

$$C_{ijk}(x, y) := \frac{1}{4}\left[F^2\right]_{y^i y^j y^k} = \frac{1}{2}\frac{\partial g_{ij}(x, y)}{\partial y^k}.$$

It is clear that a Finsler metric F is Riemann metric if and only if $C_{ijk} = 0$. Further, let us consider the Busemann–Hausdorff volume form (see [19])

$$dV_{BH} := \sigma_{BH}(x) \omega^1 \wedge \cdots \wedge \omega^n,$$

where

$$\sigma_{BH}(x) := \frac{\omega_n}{\mathrm{Vol}\{(y^i) \in R^n \mid F(x, y) < 1\}}$$

and ω_n denotes the volume of the unit ball in R^n and $\mathrm{Vol}\{\cdot\}$ denotes the Euclidean volume function on subsets in R^n.

Example 1.1. If F is a Riemannian metric $F = \sqrt{g_{ij}(x) y^i y^j}$, then

$$\sigma_{BH}(x) = \sqrt{\det\bigl(g_{ij}(x)\bigr)}.$$

Take arbitrary standard local coordinate system (x^i, y^i) in TM and let

$$\tau(x, y) := \ln\left[\frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma_{BH}(x)}\right]. \tag{6}$$

τ is called the *distortion* of F. Distortion τ characterizes the Riemann metrics among Finsler metrics. In fact, F is Riemannian if and only if $\tau = 0$ [12].

It is natural to study the rate of change of the distortion along geodesics. Let

$$\mathbf{S}(x, y) := \tau_{;m}(x, y) y^m, \tag{7}$$

where "$;$" denotes the horizontal covariant derivative with respect to F. $\mathbf{S}$ is called the *S-curvature* of F [20].

Remark 1.3. Suppose that $\tilde{\sigma} = \tilde{\sigma}(x)$ is another volume coefficient. We can write $\sigma_{BH}(x) = \rho(x)\tilde{\sigma}(x)$. Then

$$\mathbf{S} = \widetilde{\mathbf{S}} + \eta,$$

where $\eta := -d(\ln\rho)(y)$ is a closed 1-form.

A Finsler metric F on an n-dimensional manifold M is said to be of *isotropic S-curvature* if there exists a scalar function $c(x)$ on M such that

$$\mathbf{S}(x, y) = (n+1)c(x)F(x, y).$$

If $c(x) = \text{constant}$, we say that F is of *constant S-curvature*.

A fundamental result for S-curvature is that, for any Berwald metric, $\mathbf{S} = 0$ [20]. In particular, $\mathbf{S} = 0$ for Riemannian metrics.

2. Locally dually flat Finsler metrics

The notion of dually flat metrics was first introduced by S.-I. Amari and H. Nagaoka [2] when they study the information geometry on Riemannian spaces. Later on, Z. Shen consider the information geometry on Finsler manifolds [22]. Briefly, information geometry emerges from investigating the geometrical structure of a family of probability distributions and has been applied successfully to various areas including statistical inference, control system theory and multi-terminal information theory.

Locally dually flat Finsler metrics are studied in Finsler information geometry and naturally arise from the investigation of the so-called flat information structure [22]. A Finsler metric $F = F(x, y)$ on a manifold is *locally dually flat* if at every point there is a coordinate system (x^i) in which the geodesic spray coefficients are in the following form

$$G^i = -\frac{1}{2}g^{ij}H_{y^j}, \tag{8}$$

where $H = H(x, y)$ is a C^∞ scalar function on $TM \setminus \{0\}$ satisfying $H(x, \lambda y) = \lambda^3 H(x, y)$ for all $\lambda > 0$. Such a coordinate system is called an *adapted coordinate system*. Dually flat Finsler metrics on an open subset in R^n can be characterized by a simple PDE.

Lemma 2.1. *(See [22].) A Finsler metric $F = F(x, y)$ on an open subset $\mathcal{U} \subset R^n$ is locally dually flat if and only if it satisfies the following equations:*

$$[F^2]_{x^k y^l} y^k - 2[F^2]_{x^l} = 0. \tag{9}$$

In this case, $H = H(x, y)$ in (8) is given by $H = -\frac{1}{6}[F^2]_{x^m} y^m$.

It is known that a Riemannian metric $F = \sqrt{g_{ij}(x)y^i y^j}$ is locally dually flat if and only if in an adapted coordinate system,

$$g_{ij}(x) = \frac{\partial^2\psi}{\partial x^i \partial x^j}(x),$$

where $\psi = \psi(x)$ is a C^∞ function [1,2].

The first example of non-Riemannian dually flat metrics is given in [22] as follows.

$$F = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - |x|^2} \pm \frac{\langle x, y\rangle}{1 - |x|^2}. \tag{10}$$

This metric is just the Funk metric defined on the unit ball $B^n \subset R^n$.

The Finsler metric in (10) is also locally projectively flat with constant flag curvature $\mathbf{K} = -\frac{1}{4}$ and $P = (1/2)Fy^i$. More general, we have the following

Example 2.1. Let $\mathcal{U} \subset R^n$ be a strongly convex domain, namely, there is a Minkowski norm $\phi(y)$ on R^n such that

$$\mathcal{U} := \{y \in R^n \mid \phi(y) < 1\}.$$

Define $\Theta = \Theta(x, y) > 0$, $y \neq 0$ by

$$x + \frac{y}{\Theta} \in \partial\mathcal{U}, \qquad y \in T_x\mathcal{U} = R^n.$$

It is easy to show that Θ is a Finsler metric satisfying

$$\Theta_{x^k} = \Theta \Theta_{y^k}. \tag{11}$$

Using (11), one can easily verify that $\Theta = \Theta(x, y)$ satisfies (9) and (3). Thus it is locally dually flat and projectively flat on $\mathcal{U}$. Θ is called the *Funk metric* on $\mathcal{U}$. It is easy to see that Funk metric is of constant flag curvature $\mathbf{K} = -\frac{1}{4}$. Also, Θ is of constant S-curvature, $\mathbf{S} = \frac{n+1}{2}\Theta$. In particular, when $\mathcal{U} = B^n(1)$, the Funk metric is just the metric in the form of (10).

In fact, every locally dually flat and projectively flat metric on an open subset in R^n must be either a Minkowski metric or a Funk metric satisfying (11) after a normalization.

Theorem 2.2. *(See [11].) Let $F = F(x, y)$ be a Finsler metric on an open subset $\mathcal{U} \subset R^n$. Then F is locally dually flat and projectively flat on $\mathcal{U}$ if and only if F satisfies the following*

$$F_{x^k} = CFF_{y^k}, \tag{12}$$

where C is a constant. In this case, F is of constant flag curvature $\mathbf{K} = -\frac{1}{4}C^2$.

Proof. Suppose that (3) and (9) hold, that is, we have the following

$$[F^2]_{x^k y^l} y^k - 2[F^2]_{x^l} = 0, \qquad F_{x^k y^l} y^k - F_{x^l} = 0. \tag{13}$$

From (13), we can prove that $F_{x^k} = 2PF_{y^k}$, where $P := F_{x^m} y^m/(2F)$ is projective factor. Further, we can obtain $P = \frac{1}{2}CF$ and $C =$ constant. Hence, by (5), we get

$$\mathbf{K} = -\frac{1}{4}C^2.$$

The converse is clear. □

Obviously, Theorem 2.2 generalizes enormously the result on Funk metric in Example 2.1.

3. Locally dually flat Randers metrics

Randers metrics form an important and ubiquitous class of Finsler metrics with a strong presence in both the theory and applications of Finsler geometry, and studying Randers metrics is an important step to understand general Finsler metrics. In 1974, M. Matsumoto proved that Randers metric $F = \alpha + \beta$ is a Landsberg metric if and only if β is parallel with respect to α [16]. Later, S. Kikuchi proved that Randers metric $F = \alpha + \beta$ is a Berwald metric if and only if β is parallel with respect to α [14]. He also proved that Randers metric $F = \alpha + \beta$ is locally Minkowskian if and only if α is flat and β is parallel with respect to α. In 2003, Z. Shen classified projectively flat Randers metrics of constant flag curvature [21]. At the same time, D. Bao and C. Robles proved that, if Randers metric F is Einstein with $\mathbf{Ric} = (n-1)\mathbf{K}(x)F^2$, then F is of constant S-curvature [4]. Hence, it is natural to consider projectively flat Randers metrics with isotropic S-curvature. In [7], X. Mo, Z. Shen and the first author classified projectively flat Randers metrics with isotropic S-curvature. On the other hand, it is important to characterize the Randers metrics of constant flag curvature (see [4,17,25]). In 2004, D. Bao, C. Robls and Z. Shen classified Randers metrics of constant flag curvature [5]. Further, Z. Shen and the author classified Randers metrics of scalar flag curvature with isotropic S-curvature [9]. This class of Randers metrics contains all projectively flat Randers metrics with isotropic S-curvature and Randers metrics of constant flag curvature.

Now, let us consider the locally dually flat Randers metrics. Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M. Define $b_{i|j}$ by

$$b_{i|j}\theta^j := db_i - b_j\theta_i^j,$$

where "|" denotes the covariant derivative with respect to α. Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \qquad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}), \qquad s^i_j := a^{ih}s_{hj},$$
$$s_j := b^i s_{ij}, \qquad r_j := b^i r_{ij}, \qquad e_{ij} := r_{ij} + b_i s_j + b_j s_i.$$

First we have the following identities

$$\alpha_{x^k} = \frac{y_m}{\alpha}\frac{\partial G^m_\alpha}{\partial y^k}, \qquad \beta_{x^k} = b_{m|k}y^m + b_m\frac{\partial G^m_\alpha}{\partial y^k},$$
$$s_{y^k} = \frac{\alpha b_k - s y_k}{\alpha^2},$$

where $s := \beta/\alpha$ and $y_k := a_{jk}y^j$.

By (9), a Randers metric $F=\alpha+\beta$ is locally dually flat if and only if α and β satisfy the following

$$\frac{\alpha^2 b_k-\beta y_k}{\alpha^3}\big[2(y_m+\alpha b_m)G^m_\alpha+\alpha r_{00}\big] \\ +(1+s)\left\{2\left(a_{mk}+\frac{y_k}{\alpha}b_m\right)G^m_\alpha-(y_m+\alpha b_m)\frac{\partial G^m_\alpha}{\partial y^k}+\frac{r_{00}}{\alpha}y_k+\alpha(3s_{k0}-r_{k0})\right\}=0. \tag{14}$$

Multiplying (14) by α^3 yields

$$\big(b_k\alpha^2-\beta y_k\big)\big[2(y_m+\alpha b_m)G^m_\alpha+\alpha r_{00}\big] \\ +(\alpha+\beta)\alpha\left[2(a_{mk}\alpha+y_k b_m)G^m_\alpha-\big(\alpha y_m+\alpha^2 b_m\big)\frac{\partial G^m_\alpha}{\partial y^k}+r_{00}y_k+\alpha^2(3s_{k0}-r_{k0})\right]=0. \tag{15}$$

Rewriting (15) as a polynomial in α, we have

$$\left(-b_m\frac{\partial G^m_\alpha}{\partial y^k}+3s_{k0}-r_{k0}\right)\alpha^4+\left[2b_k b_m G^m_\alpha+b_k r_{00}+2a_{mk}G^m_\alpha-y_m\frac{\partial G^m_\alpha}{\partial y^k}-\beta b_m\frac{\partial G^m_\alpha}{\partial y^k}+\beta(3s_{k0}-r_{k0})\right]\alpha^3 \\ +\left(2b_k y_m G^m_\alpha+2y_k b_m G^m_\alpha+r_{00}y_k+2\beta a_{mk}G^m_\alpha-\beta y_m\frac{\partial G^m_\alpha}{\partial y^k}\right)\alpha^2-2\beta y_k y_m G^m_\alpha=0. \tag{16}$$

From (16) we know that the coefficients of α are zero. Hence the coefficients of α^3 must be zero too. Thus we have

$$2b_k b_m G^m_\alpha+b_k r_{00}+2a_{mk}G^m_\alpha-y_m\frac{\partial G^m_\alpha}{\partial y^k}-\beta b_m\frac{\partial G^m_\alpha}{\partial y^k}+\beta(3s_{k0}-r_{k0})=0, \tag{17}$$

$$\left(-b_m\frac{\partial G^m_\alpha}{\partial y^k}+3s_{k0}-r_{k0}\right)\alpha^4+\left(2b_k y_m G^m_\alpha+2y_k b_m G^m_\alpha+r_{00}y_k+2\beta a_{mk}G^m_\alpha-\beta y_m\frac{\partial G^m_\alpha}{\partial y^k}\right)\alpha^2 \\ -2\beta y_k y_m G^m_\alpha=0. \tag{18}$$

From these equations, we can obtain the following theorem.

Theorem 3.1. *(See [11].) Let $F=\alpha+\beta$ be a Randers metric on a manifold M. F is locally dually flat if and only if in an adapted coordinate system, β and α satisfy*

$$r_{00}=\frac{2}{3}\theta\beta-\frac{5}{3}\tau\beta^2+\left[\tau+\frac{2}{3}\big(\tau b^2-b_m\theta^m\big)\right]\alpha^2, \tag{19}$$

$$s_{k0}=-\frac{\theta b_k-\beta\theta_k}{3}, \tag{20}$$

$$G^m_\alpha=\frac{1}{3}(2\theta+\tau\beta)y^m-\frac{1}{3}\big(\tau b^m-\theta^m\big)\alpha^2, \tag{21}$$

where $\tau=\tau(x)$ is a scalar function and $\theta=\theta_k y^k$ is a 1-form on M and $\theta^m:=a^{im}\theta_i$.

4. Locally dually flat Randers metrics with weak isotropic flag curvature

If a locally dually flat Randers metric is of weak isotropic flag curvature, then it can be completely determined. Firstly, from Theorem 3.1, we can prove the following

Proposition 4.1. *(See [11].) Let $F=\alpha+\beta$ be a locally dually flat Randers metric on an n-dimensional manifold M. Suppose that F is of isotropic S-curvature, $\mathbf{S}=(n+1)cF$, where $c=c(x)$ is a scalar function on M. Then F is locally projectively flat in adapted coordinate systems with $G^i=cFy^i$.*

Proof. Recall the formula for the geodesic spray coefficients G^i of F,

$$G^i=G^i_\alpha+\frac{r_{00}+2\beta s_0}{2F}y^i-s_0y^i+\alpha s^i{}_0, \tag{22}$$

where G^i_α denote the geodesic spray coefficients of α and $r_{00}:=r_{ij}y^iy^j$, $s_0:=s_jy^j$ and $s^i{}_0:=s^i{}_jy^j$.

By Theorem 3.1, α and β satisfy (19)–(21). It is shown that a Randers metric $F=\alpha+\beta$ is of isotropic S-curvature, $\mathbf{S}=(n+1)cF$, if and only if it satisfies

$$r_{00}=2c\big(\alpha^2-\beta^2\big)-2\beta s_0. \tag{23}$$

See [8]. By (19) and (23) we obtain

$$\left\{2c-\tau-\frac{2}{3}\left(\tau b^2-b_m\theta^m\right)\right\}\alpha^2=\left\{2s_0+\frac{2}{3}\theta+\left(2c-\frac{5}{3}\tau\right)\beta\right\}\beta.$$

After a series of computations, we can verify that β is closed and

$$G^i_\alpha=\tau\beta y^i=2c\beta y^i.$$

Then

$$r_{00}=2c\left(\alpha^2-\beta^2\right)$$

and α is projectively flat in the adapted coordinate system. By (22), we get

$$G^i=G^i_\alpha+\frac{r_{00}}{2F}y^i=cFy^i. \tag{24}$$

Therefore $F=\alpha+\beta$ is projectively flat in adapted coordinate systems. $\square$

In [7], the authors prove that, if a Finsler metric of scalar flag curvature is of isotropic S-curvature, $\mathbf{S}=(n+1)cF$, then the flag curvature $\mathbf{K}$ can be characterized as follows

$$\mathbf{K}=\frac{3c_{x^m}y^m}{F}+\sigma,$$

where $c=c(x)$ and $\sigma=\sigma(x)$ are scalar functions on M. Conversely, if a Randers metric $F=\alpha+\beta$ is of scalar flag curvature in the following form

$$\mathbf{K}=3c_{x^m}(x)y^m/F+\sigma(x),$$

then F is of isotropic S-curvature $\mathbf{S}=(n+1)\tilde{c}F$, $\tilde{c}(x)-c(x)=\text{constant}$ [23]. Hence, by Proposition 4.1, we have the following

Proposition 4.2. *(See [11].) Let $F=\alpha+\beta$ be a locally dually flat Randers metric on an n-dimensional manifold M. Suppose that F is of weak isotropic flag curvature,*

$$\mathbf{K}=3c_{x^m}(x)y^m/F+\sigma(x).$$

Then F is locally projectively flat in adapted coordinate systems with $G^i=\tilde{c}Fy^i$, where $\tilde{c}(x)-c(x)=\text{constant}$.

By Proposition 4.2, we obtain our classification theorem on locally dually flat Randers metrics with weak isotropic flag curvature.

Theorem 4.3. *(See [11].) Let $F=\alpha+\beta$ be a Randers metric on a manifold M. Then F is locally dually flat and is of weak isotropic flag curvature if and only if one of the following holds*:

(i) *F is locally Minkowskian.*
(ii) *α locally satisfies Hamel's projective flatness equation: $\alpha_{x^m y^k}y^m=\alpha_{x^k}$ with constant sectional curvature $\mathbf{K}_\alpha=-4c^2<0$ and $\beta=\frac{\alpha_{x^m}y^m}{4c\alpha}$. In this case, $F=\alpha+\beta$ is dually flat and locally projectively flat with constant flag curvature $\mathbf{K}=-c^2$.*

For a given constant $c\neq 0$, there might be many forms for α satisfying Hamel's projective flatness equation with constant sectional curvature $\mathbf{K}_\alpha=-4c^2$ and $\beta=\frac{\alpha_{x^m}y^m}{4c\alpha}$. Note that if we take $c=\pm\frac{1}{2}$ and

$$\alpha=\frac{\sqrt{|y|^2-(|x|^2|y|^2-\langle x,y\rangle^2)}}{1-|x|^2},$$

then

$$\beta=\pm\frac{\langle x,y\rangle}{1-|x|^2}.$$

In this case, F is just the Funk metric on the unit ball $B^n\subset R^n$ given in (10).

5. A class of locally dually flat (α, β)-metrics

In Riemann–Finsler geometry, (α,β)-metrics form an important class of Finsler metrics which are defined by a Riemann metric $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and a 1-form $\beta = b_i y^i$ on an n-dimensional manifold M. They are expressed in the form

$$F = \alpha\phi(s), \quad s = \beta/\alpha,$$

where $\phi(s)$ is a C^∞ positive function on $(-b_o, b_o)$. It is known that $F = \alpha\phi(\beta/\alpha)$ is a positive definite Finsler metric for any α and β with $b := \|\beta_x\|_\alpha < b_o$ if and only if ϕ satisfies the following condition (see [12]):

$$\phi(s) - s\phi'(s) + \left(b^2 - s^2\right)\phi''(s) > 0 \quad \left(|s| \leqslant b < b_o\right). \tag{25}$$

Such a metric is called an (α,β)-*metric*. In particular, when $\phi = 1 + s$, the Finsler metric $F = \alpha + \beta$ is Randers metric with $\|\beta_x\|_\alpha < 1$.

In this section, we mainly consider (α,β)-metrics in the following form

$$F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}, \tag{26}$$

where ϵ, k $(\neq 0)$ are constants. In this case, $\phi = 1 + \epsilon s + ks^2$. Let $b_0 = b_0(\epsilon, k) > 0$ be the largest number such that

$$\phi = 1 + \epsilon s + ks^2 > 0, \qquad 1 + 2kb^2 - 3ks^2 > 0, \qquad |s| \leqslant b < b_0.$$

Then F is a Finsler metric if and only if $b := \|\beta_x\|_\alpha < b_o$ for any $x \in M$. In particular,

$$F = \frac{(\alpha+\beta)^2}{\alpha} \tag{27}$$

is a Finsler metric if and only if $\|\beta_x\|_\alpha < 1$. A famous example of the metrics in the form of (27) is Berwald's metric which is defined by

$$B = \frac{(\alpha+\beta)^2}{\alpha}, \qquad y \in T_x\mathbf{B}^n,$$

where $\alpha = \lambda\bar{\alpha}$, $\beta = \lambda\bar{\beta}$ and

$$\bar{\alpha} = \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}}{1-|x|^2}, \qquad \bar{\beta} = \frac{\langle x, y\rangle}{1-|x|^2}, \qquad \lambda = \frac{1}{1-|x|^2}.$$

Berwald's metric is a projectively flat (α,β)-metric with $\mathbf{K} = 0$. More generally, when $n \geqslant 3$ and under certain extra conditions, B. Li and Z. Shen prove that an (α,β)-metric $F = \alpha\phi(\beta/\alpha)$ is projectively flat with constant flag curvature $\mathbf{K}$ if and only if one of the following holds [15].

(1) α is projectively flat and β is parallel with respect to α. In this case, F is a projectively flat Berwald metric.
(2) $\phi = \sqrt{1+ks^2} + \epsilon s$, where k, ϵ $(\neq 0)$ are constants. In this case, $F = \bar{\alpha} + \bar{\beta}$ is a Randers metric with $\mathbf{K} < 0$, where

$$\bar{\alpha} = \sqrt{\alpha^2 + k\beta^2}, \qquad \bar{\beta} = \epsilon\beta.$$

(3) $\phi = (\sqrt{1+ks^2} + \epsilon s)^2/\sqrt{1+ks^2}$, where k, ϵ $(\neq 0)$ are constants. In this case, $F = (\bar{\alpha}+\bar{\beta})^2/\bar{\alpha}$ with $\mathbf{K} = 0$, where

$$\bar{\alpha} = \sqrt{\alpha^2 + k\beta^2}, \qquad \bar{\beta} = \epsilon\beta.$$

Hence, it is natural to study Randers metrics and the (α,β)-metrics in the form of (27).

Motivated by the research in [11], Q. Xia proves the following

Theorem 5.1. *(See [24].) Let $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ be a Finsler metric on a manifold M, where ϵ, k are nonzero constants. Then F is locally dually flat if and only if in an adapted coordinate system, α and β satisfy*

$$r_{00} = \frac{2}{3}\left[\theta\beta - \left(b_m\theta^m\right)\alpha^2\right],$$

$$s_{k0} = -\frac{\theta b_k - \beta\theta_k}{3},$$

$$G^m_\alpha = \frac{1}{3}\left(2\theta y^m + \theta^m\alpha^2\right),$$

where $\theta = \theta_k y^k$ is a 1-form on M and $\theta^m := a^{im}\theta_i$.

In [10], we characterize completely (α,β)-metrics of non-Randers type with isotropic S-curvature. Based on the main result in [10] and by use of Theorem 5.1, we can prove the following

Proposition 5.2. *(See [24].) Let $F=\alpha+\epsilon\beta+k\frac{\beta^2}{\alpha}$ be a Finsler metric on a manifold M, where ϵ, k are nonzero constants. Then F is locally dually flat with isotropic S-curvature if and only if α is flat and β is parallel with respect to α. In this case, F is locally isometric to the following Minkowski metric*

$$\tilde{F}=|y|+\epsilon\left(b_i y^i\right)+k\frac{(b_i y^i)^2}{|y|},$$

where $|\cdot|$ is the Euclidean metric on R^n and b_i $(1\leqslant i\leqslant n)$ are constants.

Further, we can characterize locally dually flat (α,β)-metrics in the form of (26) of scalar flag curvature.

Theorem 5.3. *(See [24].) Let $F=\alpha+\epsilon\beta+k\frac{\beta^2}{\alpha}$ be a Finsler metric on a manifold M of dimension n $(n\geqslant 2)$, where ϵ, k are nonzero constants. Then F is locally dually flat Finsler metric of scalar flag curvature if and only if $\epsilon^2=4k$ and α is flat and β is parallel with respect to α. In this case, F is locally isometric to the following Minkowski metric*

$$\bar{F}=\frac{(|y|+\sqrt{k}(b_i y^i))^2}{|y|},$$

where $|\cdot|$ is the Euclidean metric on R^n and b_i $(1\leqslant i\leqslant n)$ are constants.

References

[1] S.-I. Amari, Differential-Geometrical Methods in Statistics, Springer Lecture Notes in Statistics, vol. 28, Springer-Verlag, 1985.
[2] S.-I. Amari, H. Nagaoka, Methods of Information Geometry, AMS Translation of Math. Monographs, vol. 191, Oxford University Press, 2000.
[3] S. Bácsó, M. Matsumoto, On Finsler spaces of Douglas type. A generalization of the notion of Berwald space, Publ. Math. Debrecen 51 (1997) 385–406.
[4] D. Bao, C. Robles, On Randers spaces of constant flag curvature, Rep. Math. Phys. 51 (2003) 9–42.
[5] D. Bao, C. Robles, Z. Shen, Zermelo navigation on Riemannian manifolds, J. Diff. Geom. 66 (3) (2004) 377–435.
[6] D. Bao, Z. Shen, Finsler metrics of constant positive curvature on the Lie group S^3, J. London Math. Soc. 66 (2002) 453–467.
[7] X. Cheng, X. Mo, Z. Shen, On the flag curvature of Finsler metrics of scalar curvature, J. London Math. Soc. 68 (2) (2003) 762–780.
[8] X. Chen(g), Z. Shen, Randers metrics with special curvature properties, Osaka J. Math. 40 (2003) 87–101.
[9] X. Cheng, Z. Shen, Randers metrics of scalar flag curvature, J. Austral. Math. Soc. 87 (2009) 359–370.
[10] X. Cheng, Z. Shen, A class of Finsler metrics with isotropic S-curvature, Israel J. Math. 169 (1) (2009) 317–340.
[11] X. Cheng, Z. Shen, Y. Zhou, On a class of locally dually flat Finsler metrics, Int. J. Math. 21 (11) (2010) 1531–1543.
[12] S.S. Chern, Z. Shen, Riemann–Finsler Geometry, Nankai Tracts in Mathematics, vol. 6, World Scientific, 2005.
[13] G. Hamel, Über die Geometrieen in denen die Geraden die Kürzesten sind, Math. Ann. 57 (1903) 231–264.
[14] S. Kikuchi, On the condition that a space with (α,β)-metric be locally Minkowskian, Tensor (N.S.) 33 (1979) 242–246.
[15] B. Li, Z. Shen, On a class of projectively flat Finsler metrics with constant flag curvature, Int. J. Math. 18 (2007) 749–760.
[16] M. Matsumoto, On Finsler spaces with Randers metric and special forms of important tensors, J. Math. Kyoto Univ. 14 (1974) 477–498.
[17] M. Matsumoto, H. Shimada, The corrected fundamental theorem on Randers spaces of constant curvature, Tensor (N.S.) 63 (2002) 43–47.
[18] G. Randers, On an asymmetrical metric in the four-space of general relativity, Phys. Rev. 59 (1941) 195–199.
[19] Z. Shen, Volume comparison and its applications in Riemann–Finsler geometry, Adv. Math. 128 (1997) 306–328.
[20] Z. Shen, Differential Geometry of Spray and Finsler Spaces, Kluwer Academic Publishers, 2001.
[21] Z. Shen, Projectively flat Randers metrics with constant flag curvature, Math. Ann. 325 (1) (2003) 19–30.
[22] Z. Shen, Riemann–Finsler geometry with applications to information geometry, Chin. Ann. Math. B 27 (1) (2006) 73–94.
[23] Z. Shen, G.C. Yildirim, A characterization of Randers metrics of scalar flag curvature, preprint, 2008.
[24] Q. Xia, On a class of locally dually flat Finsler metrics, Publ. Math. Debrecen 78 (1) (2011) 169–190.
[25] H. Yasuda, H. Shimada, On Randers spaces of scalar curvature, Rep. Math. Phys. 11 (1977) 347–360.